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Small Oscillations of a Viscous Isothermal Atmosphere*

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The downward reflection of acoustic-gravity waves produced in an isothermal atmosphere by the exponential increase with altitude of the kinematic viscosity is investigated. The problem is more general than those previously considered in [1, 2]. The ordinary differential equations for harmonic oscillations were integrated numerically by a modification of a procedure due to Conte [6], and the dependence of the reflection coefficient on the horizontal wave number k and the vertical wave number β was obtained. It was found that, in contrast to the results for a stratified inviscid fluid [1], the magnitude of the reflection coefficient depends on k. The deviation from previous results is greatest when k and β are of the same order of magnitude and not too large or too small.

I. INTRODUCTION

It is well known that acoustic-gravity waves which travel upward in a stratified "atmosphere" may be reflected downward if the mean temperature varies with height. They may also be reflected if dissipative forces are present, provided these increase rapidly with height z. This was demonstrated in [1] by means of a simple model of two-dimensional waves in an incompressible stratified fluid with exponentially decreasing density $\rho(z)$ and constant dynamic viscosity μ (i.e., with exponentially increasing kinematic viscosity $\mu/\rho(z)$). It was found that the magnitude of the reflection coefficient $|K_R|$ tends to $\exp(-2\pi^2 H/L)$ as $\mu \to 0$ (H is the density scale height, L the vertical wave length). This result contains the somewhat surprising feature that the reflection does not depend on the horizontal wave length.

A similar problem for a viscous isothermal atmosphere was considered in [2],

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but only for vertically propagating acoustic waves, and the same asymptotic formula for the reflection coefficient was found to hold (see also [3]). On the other hand, Lindzen showed in [4] that this same result is valid for atmospheric tidal waves when dissipation is introduced through the mechanism of Newtonian cooling rather than through viscosity (see also [5]). This led him to the conjecture that the asymptotic value of $|K_R|$ is independent of the details of the fluid model. Our work will show that this is not always the case.

We will consider small two-dimensional oscillations in a compressible isothermal atmosphere with a constant dynamic viscosity coefficient. This problem is analytically considerably more complicated than the ones referred to above, and we have, therefore, resorted to a numerical study. The numerical problem itself is not trivial since it requires the computation of solutions of a system of differential equations which is inherently unstable, i.e., one which possesses solutions with widely differing rates of growth. The method employed here is a modification of a procedure due to Conte [6]. It will be described briefly in Section 3; a more complete description and an error analysis is given in [7].

It was found that $|K_R| \leqslant \exp(-2\pi^2 H/L)$, and that it depends on the horizontal scale of the motion. The greatest deviation from $\exp(-2\pi^2 H/L)$ takes place when the horizontal and vertical wave lengths are comparable to each other, while for large and small ratios of these lengths the previously obtained asymptotic formula is accurate. This is consistent with the results in [2] and [4]. The deviation was found to be much larger for acoustic waves than for gravity waves, and for some cases $|K_R|$ was not even a monotonic function of H/L.

II. Formulation of the Problem

We will consider small oscillations of a viscous, thermally nonconducting fluid about a state of hydrostatic equilibrium with uniform temperature. Let the quantities which characterize the equilibrium state be denoted by the subscript 0. Then,

$$T_0 \equiv \text{const}, \qquad \rho_0(z) = \rho_0(0)\, e^{-z/H}, \qquad p_0(z) = gH\rho_0(z), \tag{1}$$

where $H = RT_0/g$ is the density scale height and $0 \leqslant z < \infty$. The linearized equations for the perturbed quantities are[1]:

$$\rho_0 u_t + p_x = \mu \Delta u + \tfrac{1}{3}\mu(\operatorname{div} \mathbf{v})_x, \tag{2a}$$

$$\rho_0 w_t + p_z + g\rho = \mu \Delta w + \tfrac{1}{3}\mu(\operatorname{div} \mathbf{v})_z, \tag{2b}$$

$$\rho_t + \rho_0(\operatorname{div} \mathbf{v}) + w\rho_0' = 0, \tag{2c}$$

$$p_t + c^2\rho_0 \operatorname{div} \mathbf{v} - g\rho_0 w = 0, \tag{2d}$$

[1] See Table I.

TABLE I

Table of Symbols

x, z	Horizontal and vertical cordinates
$\mathbf{v} = [u, w]$	Velocity vector, with horizontal component u and vertical component w
U, W	Oscillation amplitudes, defined in (4)
$\mathbf{y}$	$= [U, W]$
p	Pressure
ρ	Density
T	Temperature
R	Gas constant
H	Density scale height
γ	Ratio of specific heats
μ	Dynamic viscosity coefficient
c	Speed of sound
N	Brunt–Väisälä frequency
k	Horizontal wave number
β	Vertical wave number
σ	Frequency
Δ	$= \partial^2/\partial x^2 + \partial^2/\partial z^2$
ϵ	Dimensionless parameter defined in (5)
α, λ	Parameters defined in (9)
ξ	$= e^{-z}/i\epsilon\sigma$
θ	$= \xi(d/d\xi)$

Partial derivatives are denoted by subscripts.

where the prime denotes differentiation with respect to z, and the dynamic viscosity coefficient μ is assumed to be a small constant. Eliminating p and ρ yields two equations for the velocity components u and w:

$$\rho_0\{u_{tt} + gw_x - c^2(\operatorname{div}\mathbf{v})_x\} = \mu[\Delta u + \tfrac{1}{3}\operatorname{div}\mathbf{v}]_t, \tag{3a}$$

$$\rho_0\{w_{tt} + gw_z - c^2(\operatorname{div}\mathbf{v})_z + (N^2c^2/g)\operatorname{div}\mathbf{v}\} = \mu[\Delta u + \tfrac{1}{3}\operatorname{div}\mathbf{v}]_t, \tag{3b}$$

where N is the Brunt–Väisälä frequency, defined by

$$\frac{N^2}{g} = -\left(\frac{\rho_0'}{\rho_0} + \frac{g}{c^2}\right) = \frac{\gamma - 1}{\gamma}.$$

Letting

$$\begin{aligned} u(x, z, t) &= U(z)\exp i(kx - \sigma t), \\ w(x, z, t) &= iW(z)\exp i(kx - \sigma t), \end{aligned} \tag{4}$$

and introducing the dimensionless quantities

$$\tilde{z} = z/H, \qquad \tilde{x} = x/H, \qquad \tilde{k} = kH, \qquad \tilde{\sigma} = \sigma(H/g)^{1/2},$$
$$\tilde{\rho}(z) = \rho_0(z)/\rho_0(0), \qquad \epsilon = \frac{\mu}{\gamma\rho_0(0)\,H^2}(H/g)^{1/2}, \tag{5}$$

one obtains a system of ordinary differential equations:

$$A\mathbf{y}'' + B\mathbf{y}' + C\mathbf{y} = 0, \tag{6}$$

where $\mathbf{y}$ is the vector with components U and W and

$$A = \begin{bmatrix} i\epsilon\sigma & 0 \\ 0 & \rho - i\frac{4}{3}\epsilon\sigma \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & k\left(\rho - \dfrac{i\epsilon\sigma}{3}\right) \\ k\left(\rho - \dfrac{i\epsilon\sigma}{3}\right) & -\rho \end{bmatrix}, \tag{7}$$

$$C = \begin{bmatrix} \rho\left(k^2 - \dfrac{\sigma^2}{\gamma}\right) - i\epsilon\sigma\frac{4}{3}k^2 & -k\,\rho/\gamma \\ -k\,\dfrac{\gamma - 1}{\gamma}\,\rho & \dfrac{\rho\sigma^2}{\gamma} + i\epsilon\sigma k^2 \end{bmatrix}.$$

Here $\rho = e^{-z}$ and the tilde has been omitted since only the dimensionless quantities will be considered from now on.

It will be assumed for simplicity that the motion is excited by an oscillation of the boundary at $z = 0$, which results in the (normalized) boundary condition

$$U(0) = 0, \qquad W(0) = 1. \tag{8}$$

For sufficiently small values of the parameter ϵ the viscous terms in (6) are negligible in any finite interval outside of a thin boundary layer near $z = 0$. Thus, any solution $\mathbf{y}(z, \epsilon)$ of (6) behaves approximately like *some* solution $\mathbf{y}_{\text{INV}}(z)$ of the inviscid problem in any interval $[z_1, z_2]$, where $0 < z_1 < z_2 < \infty$. More precisely, $\mathbf{y}(z, \epsilon) \to \mathbf{y}_{\text{INV}}(z)$ uniformly in $[z_1, z_2]$ as $\epsilon \to 0$. The boundary layer serves to reduce $U(z)$ to zero at the boundary, and has a negligible effect on $W(z)$. Thus, the exact nature of the forcing mechanism is largely unimportant in this study, since it is concerned with reflection from the upper layers.

As is well known, the inviscid problem (obtained by setting $\epsilon = 0$ in (6)) has two exponential solutions, $\exp(\lambda_1 z)$ and $\exp(\lambda_2 z)$, with

$$\lambda_1 = \tfrac{1}{2} + (\tfrac{1}{4} - \alpha)^{1/2}, \qquad \lambda_2 = \tfrac{1}{2} - (\tfrac{1}{4} - \alpha)^{1/2}, \qquad \text{if} \quad \alpha < \tfrac{1}{4}, \tag{9a}$$

$$\lambda_1 = \tfrac{1}{2} + i\beta, \qquad \lambda_2 = \tfrac{1}{2} - i\beta, \qquad \text{if} \quad \alpha > \tfrac{1}{4}, \tag{9b}$$

where

$$\alpha = \left(\frac{\sigma^2}{\gamma} - k^2\right) + \frac{\gamma - 1}{\gamma} \frac{k^2}{\sigma^2} \quad \text{and} \quad \beta = \left(\alpha - \frac{1}{4}\right)^{1/2}.$$

If $\alpha = \frac{1}{4}$ there are two solutions of the form

$$e^{z/2} \quad \text{and} \quad ze^{z/2}. \tag{9c}$$

In cases (9a) and (9c), solutions correspond to waves which propagate horizontally. Corresponding to (9b) one has solutions of the form

$$e^{z/2}[a_1 \exp(kx + \beta z - \sigma t) + a_2 \exp(kx - \beta z - \sigma t)], \tag{10}$$

which represent an incident and a reflected wave, the reflection being produced in the region where the viscous terms in (6) are no longer negligible. One of the objects of this investigation is to compute the reflection coefficient (i.e., the ratio of the two complex amplitudes) as a function of the wave parameters σ and k. Another solution of interest is the Lamb wave [8]:

$$\sigma = \sqrt{\gamma}\, k, \qquad \mathbf{y} = \begin{bmatrix}1\\0\end{bmatrix} \exp[(\gamma - 1)\, z/\gamma], \tag{11}$$

which represents a free oscillation in the presence of a rigid boundary at $z = 0$, since $W(0) = 0$. The situation is represented schematically in Fig. 1, where the two shaded regions correspond to the travelling wave solutions of case (9b). Although these designations are somewhat arbitrary and inaccurate, we will refer to the solutions in the region marked A as acoustic waves, those from the region G as gravity waves.

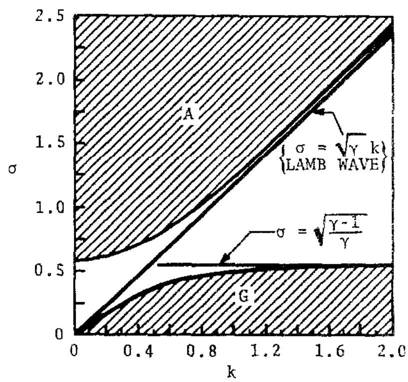

FIG. 1. Inviscid dispersion relation for $\gamma = 1.4$.

To complete the formulation it will be assumed that physically relevant solutions satisfy the "dissipation condition" (DC) that the average rate of energy dissipation in an infinite column of fluid ($0 \leqslant z < \infty$) of finite cross section be finite. Since the dissipation function depends on the squares of the velocity gradients, an equivalent requirement is:

$$\int_0^\infty \{|\mathbf{y}|^2 + |\mathbf{y}'|^2\}\, dz < \infty. \tag{12}$$

For $\epsilon > 0$ and sufficiently large z, the terms in (6) which are multiplied by ρ are negligible. If they are set equal to zero, there are four linearly independent solutions of the resulting system which behave like e^{-kz}, ze^{-kz}, e^{kz}, and ze^{kz}. It is evident that for $k > 0$ the last two do not satisfy the DC and should be discarded. The problem is, therefore, to find the asymptotic behaviour as $\epsilon \to 0$ of solutions of the system of differential equations (6), which satisfy (8) and the DC (12).

III. Analysis

It is convenient to transform the problem by introducing a new independent variable $\xi = e^{-z}/i\epsilon\sigma$. The differential equation (6) is then transformed into

$$A_1\theta^2\mathbf{y} + B_1\theta\mathbf{y} + C_1\mathbf{y} = 0, \tag{13}$$

where

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & \xi - 4/3 \end{bmatrix}, \qquad B_1 = -\begin{bmatrix} 0 & k(\xi - 1/3) \\ k(\xi - 1/3) & -\xi \end{bmatrix},$$

$$C_1 = \begin{bmatrix} (k^2 - \sigma^2/\gamma)\,\xi - 4k^2/3 & -\dfrac{k}{\gamma}\,\xi \\ -k\,\dfrac{\gamma - 1}{\gamma}\,\xi & \dfrac{\sigma^2}{\gamma}\,\xi + k^2 \end{bmatrix}, \tag{14}$$

and $\theta = \xi\, d/d\xi$. The positive z axis is mapped into a segment with $\arg \xi = -\pi/2$. The point $z = \infty$ corresponds to $\xi = 0$ and $z = 0$ corresponds to $\xi = 1/i\epsilon\sigma$. The system (13) has regular singular points at $\xi = 0$ and $\xi = 4/3$, and an irregular singularity at $\xi = \infty$. The DC can be translated into a condition on the behaviour of solutions of (13) in the neighborhood of $\xi = 0$, while the limiting behaviour of the viscous problem in a fixed finite interval $[z_1, z_2]$ will go over into a condition on the asymptotic behaviour of a solution of (13) as $\xi \to \infty$ along the ray $\arg \xi = -\pi/2$.

In order to solve the viscous problem in the limit as $\epsilon \to 0$, it is necessary to relate the solutions of (13) about the regular singularity $\xi = 0$ to the solutions of (13) about the irregular singularity $\xi = \infty$. More precisely, we require the asymptotic developments of the solutions of (13), on the ray $\arg \xi = -\pi/2$, which satisfy the DC, (12). In this section, a numerical procedure is briefly discussed which was used successfully to integrate numerically the inherently unstable differential equation (13), and thereby determine the required asymptotic developments.

Since $\xi = 0$ is a regular singularity of (13), it is possible to develop a fundamental set of convergent expansions about this point [9, Chap. 4]. It is easily shown that the resulting solutions of (13) exhibit the scalar growths ξ^k, $(\ln \xi)\, \xi^k$, ξ^{-k}, and $(\ln \xi)\, \xi^{-k}$ as $\xi \to 0$. Only the solutions which grow as ξ^k and $(\ln \xi)\, \xi^k$ satisfy the DC, (12). Hence, imposing the DC is equivalent to eliminating two of the solutions of (13).

About the irregular singularity $\xi = \infty$ a fundamental set of formal asymptotic solutions can be developed. Standard procedures, e.g., see [10], involving the transformation of (13) are complicated since the characteristic growth rates are not asymptotically distinct, that is, four distinct exponential rates of growth of formal solutions of (13) do not exist. However, it can be shown [7, Appendix B] that there exist four solutions of (13) which exhibit the asymptotic growths $\xi^{-\lambda_1}$, $\xi^{-\lambda_2}$, $\xi^{-1/4} \exp(2\sigma \sqrt{\xi/\gamma})$, and $\xi^{-1/4} \exp(-2\sigma \sqrt{\xi/\gamma})$, where λ_1 and λ_2 are the scalars given in (9). In the appendix it is shown that the lead terms in the formal solutions with algebraic growth correspond to multiples of inviscid solutions.

For small $\epsilon > 0$, the question of existence and uniqueness of the solution of the viscous problem depends primarily on whether or not at least one of the two solutions satisfying the DC is asymptotic to a nonzero multiple of the exponentially growing solution as $\xi \to \infty$ [1, 7]. For every computed case it appears that the requisite properties are satisfied for existence and uniqueness of the solution to the viscous problem.

The solution of the viscous problem $\mathbf{y}_{VP}(\xi)$ can be represented as a linear combination of any two linearly independent solutions which satisfy the DC. In particular, $\mathbf{y}_{VP}(\xi)$ can be represented as a linear combination of $\mathbf{DC}_1(\xi)$ and $\mathbf{DC}_2(\xi)$ where

$$\mathbf{DC}_1(\xi) = \sum_{n=0}^{\infty} \mathbf{a}_k \xi^{(n+k)}, \qquad \mathbf{a}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{15}$$

$$\mathbf{DC}_2(\xi) = \sum_{n=0}^{\infty} \mathbf{b}_k \xi^{(n+k)} + (\ln \xi)\, \mathbf{DC}_1(\xi), \qquad \mathbf{b}_0 = \begin{bmatrix} 1 \\ \dfrac{k-7}{k} \end{bmatrix}. \tag{16}$$

The solutions of (13) which exhibit the asymptotic scalar growths $\xi^{-\lambda_1}$, $\xi^{-\lambda_2}$, and $\xi^{-1/4} \exp(\pm 2\sigma \sqrt{\xi/\gamma})$ are denoted by $\mathbf{y}_1(\xi)$, $\mathbf{y}_2(\xi)$, and $\mathbf{y}_\pm(\xi)$, respectively. These four solutions are linearly independent and, hence, $\mathbf{y}_{VP}(\xi)$ can also be represented as a linear combination of these solutions.

Due to the different asymptotic rates of growth of $\mathbf{y}_{1,2}(\xi)$ and $\mathbf{y}_{\pm}(\xi)$ it follows that these solutions are significant in different regions. For example, the boundary layer solution $\mathbf{y}_{+}(\xi)$ is significant only in a relatively thin boundary layer near $z = 0$ whose thickness is $o(\sqrt{\epsilon/\sigma})$ as $\epsilon \to 0$. Similarly, the solution $\mathbf{y}_{-}(\xi)$ is important only in a region which is intermediate between large ξ and small ξ or equivalently in a region where the kinematic viscosity varies from small to large values. For small ξ the solutions $\mathbf{DC}_1(\xi)$ and $\mathbf{DC}_2(\xi)$ provide the correct means of determining $\mathbf{y}_{VP}(\xi)$. For large ξ, the asymptotic developments of $\mathbf{y}_{1,2}(\xi)$ and $\mathbf{y}_{\pm}(\xi)$ are accurate approximations of these solutions. In order to solve completely the viscous problem we require an overlapping region where $\mathbf{DC}_1(\xi)$, $\mathbf{DC}_2(\xi)$, $\mathbf{y}_{1,2}(\xi)$, and $\mathbf{y}_{\pm}(\xi)$ can be accurately determined.

If such an overlapping region exists, then the viscous problem can be solved in essentially two steps for small $\epsilon > 0$. First, determine a nonzero linear combination of $\mathbf{DC}_1(\xi)$ and $\mathbf{DC}_2(\xi)$ which eliminate the boundary layer solution $\mathbf{y}_{+}(\xi)$, that is, solve

$$C_1\mathbf{DC}_1(\xi_0) + C_2\mathbf{DC}_2(\xi_0) + C_3\mathbf{y}_1(\xi_0) + C_4\mathbf{y}_{-}(\xi_0) = \mathbf{y}_2(\xi_0) \tag{17}$$

and

$$C_1\theta\mathbf{DC}_1(\xi_0) + C_2\theta\mathbf{DC}_2(\xi_0) + C_3\theta\mathbf{y}_1(\xi_0) + C_4\theta\mathbf{y}_{-}(\xi_0) = \theta\mathbf{y}_2(\xi_0) \tag{18}$$

at a fixed finite value of ξ_0. Second, solve

$$\begin{bmatrix}0\\1\end{bmatrix} = d_1\{\mathbf{y}_2(\xi_1) - C_3\mathbf{y}_1(\xi_1)\} + d_2\mathbf{y}_{+}(\xi_1) \tag{19}$$

at $\xi_1 = 1/i\epsilon\sigma$. It should be noted that the solution of the viscous problem $\mathbf{y}_{VP}(\xi)$ approximately satisfies

$$\mathbf{y}_{VP}(\xi) \approx d_1\{C_1\mathbf{DC}_1(\xi) + C_2\mathbf{DC}_2(\xi)\}. \tag{20}$$

The solution represented by (20) is modified in the boundary layer. Hence, Eqs. (17) and (20) imply that, for large ξ or equivalently small kinematic viscosity, the solution of the viscous problem can be approximated by a linear combination of inviscid solutions. We are primarily interested in the linear combination of inviscid solutions which results in the limit as $\epsilon \to 0$. The region where the solution of the viscous problem is accurately approximated by a linear combination of inviscid solutions will be designated the inviscid region.

The formal asymptotic solutions of (13) will yield accurate approximations of actual solutions of (13) for large ξ. Hence, it is possible to solve (19) after the constant C_3 has been determined in (17) and (18). However, implicit in the Eqs. (17) and (18) is the requirement that the accurate values of the formal asymptotic solutions be continued to fairly small values of ξ, or $\mathbf{DC}_1(\xi)$ and $\mathbf{DC}_2(\xi)$ should be continued to large values of ξ. If a numerical integration of (13) is considered, then it is preferable to consider the integration proceeding in the direction of decreasing $|\xi|$. It can be shown [7] that there is at most an algebraic magnification of the initial relative error if solutions of (13) are computed exactly in the direction of decreasing $|\xi|$. This can be shown despite the fact that solutions of (13) exhibit different exponential rates of growth. Since only a finite number of digits are retained in the process of computing, some care must be exercised in performing the calculations. Roughly speaking, it is necessary to append a process to the numerical integration of (13) which eliminates the possibility of ill-conditioning of a fundamental set of solutions. Frequently, as a bonus, a reasonable attempt at ensuring linear independence of such a fundamental set results in an effective control of inherent error growth, e.g., see [6, 7].

The inherent instability of (13) affects the numerical calculations in a rather peculiar manner. For example, suppose one considers the problem of calculating an approximation of $\mathbf{y}_1(\xi)$. For an integration of (13) along $\arg \xi = -\pi/2$ and in the direction of decreasing $|\xi|$, the solution $\mathbf{y}_1(\xi)$ is exponentially dominated by $\mathbf{y}_-(\xi)$, i.e., $\mathbf{y}_-(\xi)$ grows exponentially fast whereas $\mathbf{y}_1(\xi)$ exhibits algebraic growth. Hence, it is anticipated that the calculation of $\mathbf{y}_1(\xi)$ may be very difficult. However, $\mathbf{y}_1(\xi)$ is only required to have a prescribed asymptotic expansion (see the appendix). The asymptotic properties of $\mathbf{y}_1(\xi)$ are satisfied by a family of solutions of (13), where two distinct members of the family differ by a multiple of $\mathbf{y}_-(\xi)$. Arbitrary multiples of $\mathbf{y}_-(\xi)$ can be added to $\mathbf{y}_1(\xi)$ with no modification of the asymptotic properties of the resultant vector. Thus, the goal is to approximate a member of a family of solutions of (13) rather than a specific solution of (13).

The only numerical difficulty which results from the inherent instability of (13) is that the multiple of $\mathbf{y}_-(\xi)$, introduced via an error, can grow to such proportions that it masks $\mathbf{y}_1(\xi)$, i.e., several significant decimal digits are required to merely compute the useless multiple of $\mathbf{y}_-(\xi)$. In order to control the multiple of $\mathbf{y}_-(\xi)$ present in the numerical approximation of $\mathbf{y}_1(\xi)$, it is sufficient to append some process to the numerical integration of (13) which ensures that $\mathbf{y}_1(\xi)$ is linearly independent of $\mathbf{y}_-(\xi)$.

The numerical integration of (13) over a large ξ interval was carried out by breaking the large ξ interval into several smaller intervals. The numerical integration of (13) begins at a large value of ξ, where the initial vector is computed by means of an asymptotic expansion. At the end of the first subinterval, the numerical approximation of $\mathbf{y}_1(\xi)$ is forced to have a zero component corresponding to the largest component of $\mathbf{y}_-(\xi)$ by adding the proper multiple of $\mathbf{y}_-(\xi)$ to the approximation of $\mathbf{y}_1(\xi)$. The resultant vector obtained at the end of the first subinterval becomes the initial vector for a numerical integration on the second ξ interval. This process is repeated on each ξ subinterval. Similar procedures are used to calculate $\mathbf{y}_2(\xi)$.

Implicit in the algorithm outlined for the calculation of $\mathbf{y}_1(\xi)$ is the requirement that $\mathbf{y}_-(\xi)$ be accurately determined. The vector $\mathbf{y}_-(\xi)$ is easily calculated since this solution has dominant exponential growth for a numerical integration in the direction of decreasing $|\xi|$. In addition, by means of (15) and (16), it is possible to compute $\mathbf{DC}_1(\xi)$ and $\mathbf{DC}_2(\xi)$ and, hence, the viscous solution $\mathbf{y}_{PV}(\xi)$ can be determined. A more detailed description of the algorithm and an error analysis is contained in [7].

The numerical method described above is not the only one which could have been employed. A convenient version of the Gaussian elimination scheme has frequently been used for similar problems (see, e.g., [11, 5, or 3]). This procedure would be applied directly to the system of differential Eqs. (6) rather than to (13). The advantage of our method is that an error analysis and a convenient check on the growth of the error are available. The numerical integration was performed in single precision on an IBM 7094, and the results appear to be correct to four significant figures.

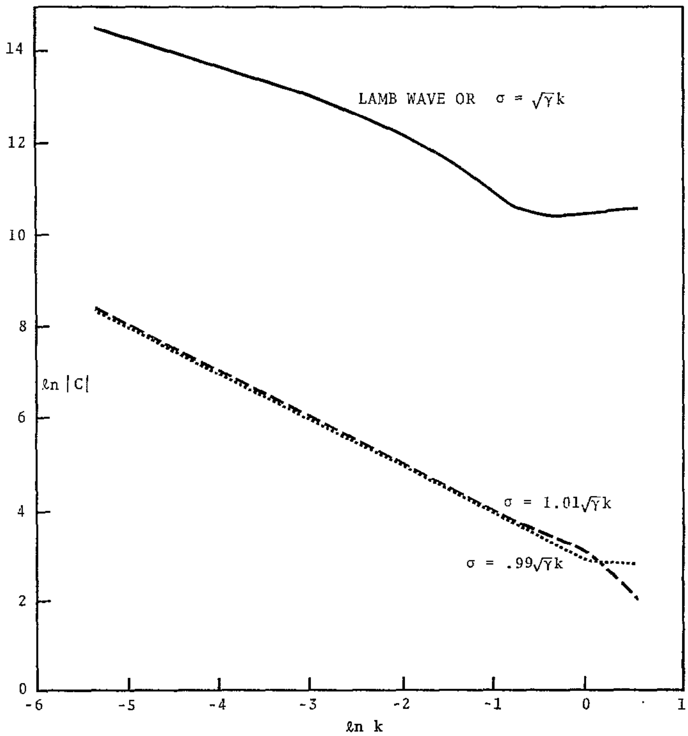

FIG. 2. ln $|C|$ at the frequency of the Lamb wave and at neighboring frequencies. C is defined in (21).

IV. Computations and Conclusions

The solution of the viscous problem for small $\epsilon > 0$ is obtained by solving Eqs. (17)–(19). The most difficult numerical problem encountered is the determination of vectors at a fixed finite value of ξ which specify the different asymptotic solutions. There are several cases which must be considered separately. The individual cases can be classified according to the character of the exponents λ_1, λ_2 in (9). We are primarily interested in those cases which modify the conclusions reached by Yanowitch [1, 2] and Lindzen [4].

Before proceeding with a summary of the calculations, it is useful to note that for the earth's atmosphere the dimensionless parameter ϵ is comparable to 10^{-11}. The dimensionless parameters k and σ, when equal to unity, correspond to a horizontal wavelength of 45 km and a frequency of 2.5 radians per minute, respectively.

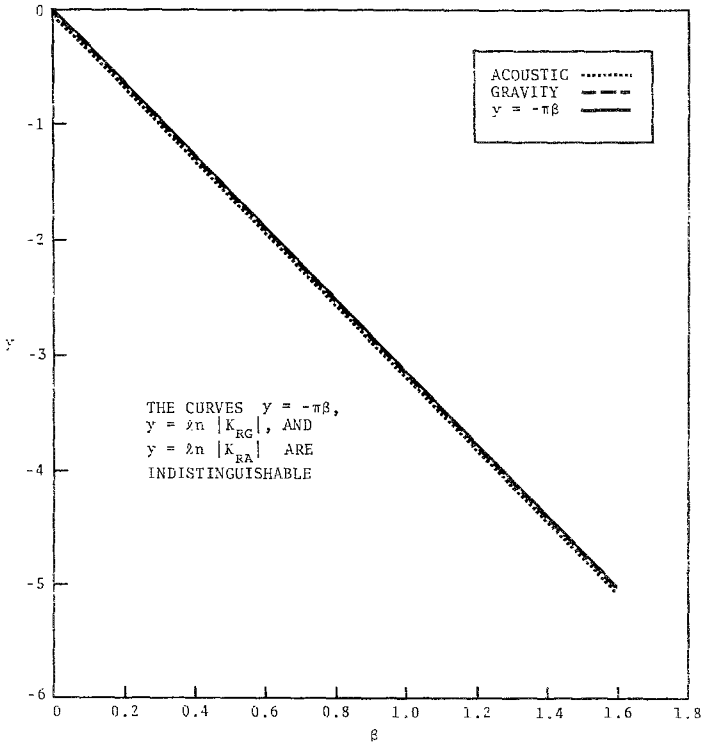

Fig. 3. Logarithm of the modulus of the reflection coefficient for $k = 0.005$ and $k = 0.05$.

Case 1. The exponents λ_1 and λ_2 are real and $(\sigma^2/\gamma - k^2) \neq 0$. In this case the solutions are nonoscillatory in the vertical direction and the results are similar to the ones in [1, 2].

Case 2. Lamb wave, $\sigma^2/\gamma - k^2 = 0$, $\lambda_1 = 1/\gamma$, $\lambda_2 = (\gamma - 1)/\gamma$. The *inviscid* problem has a free oscillation, with the solution given by (11).

For the viscous problem,

$$\mathbf{y}_{VP}(z) \approx C\{\mathbf{y}_2(z) + D\mathbf{y}_1(z)\} \tag{21}$$

in the inviscid region, and the constants C and D can be determined from d_1, C_3, and d_2 in (19). It is easy to show that

$$D = o((\epsilon\sigma)^{\lambda_1 - \lambda_2}) \qquad \text{as} \quad \epsilon \to 0, \tag{22}$$

i.e., that $D \to 0$ as $\epsilon \to 0$. Thus, $\mathbf{y}_{VP}(z) \to C\mathbf{y}_2(z)$, which shows that in the inviscid region the solution approaches the solution of the inviscid problem (the Lamb wave).

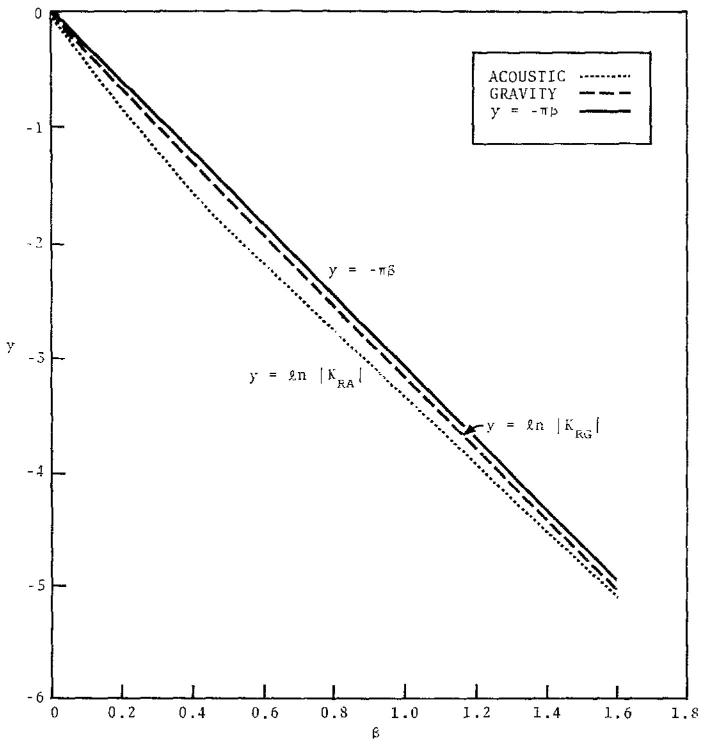

FIG. 4. Logarithm of the modulus of the reflection coefficient for $k = 0.25$.

It is to be expected that the system is resonant when ϵ is small and $\sigma^2 = \gamma k^2$. This is, in fact, the case and one can show that

$$C = o((\epsilon\sigma)^{\lambda_2 - \lambda_1}) \qquad \text{as} \quad \epsilon \to 0. \tag{23}$$

The results of the computations indicate that a change in σ of one percent from the resonant value reduces $|C|$ by a factor of about 1000 (see Fig. 2).

Case 3. The roots of the dispersion relation are complex and the inviscid solutions are wavelike in the vertical coordinate.

In the inviscid region the solution of the viscous problem can be approximated by

$$\mathbf{y}_{VP}(z) \approx A\{\mathbf{y}_2(z) + K_R \mathbf{y}_1(z)\}, \tag{24}$$

where $\mathbf{y}_1(z)$ and $\mathbf{y}_2(z)$ are inviscid solutions normalized so that the first (horizontal) component is one at $z = 0$. The solution with subscript two (one) is the inviscid solution with upward (downward) energy propagation. The scalar K_R is then defined to be the reflection coefficient.

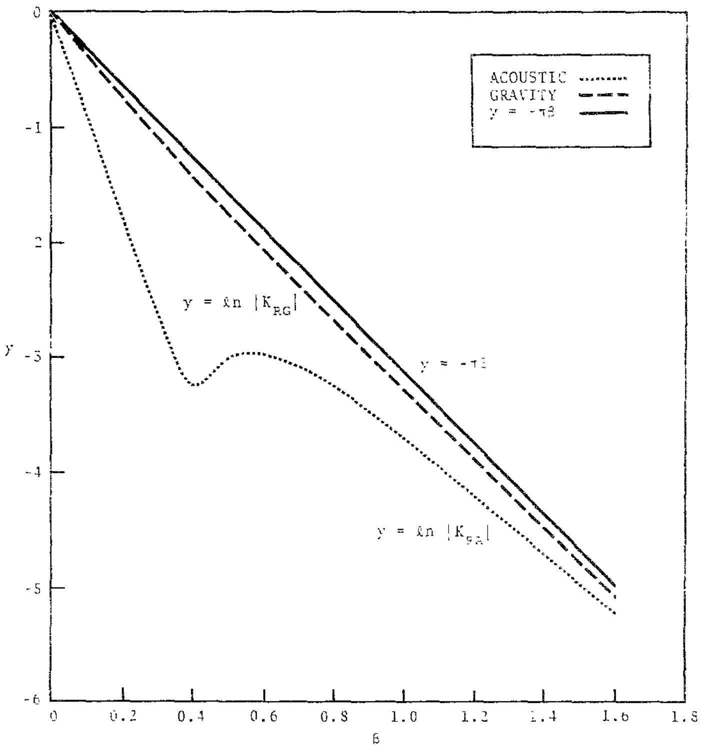

FIG. 5. Logarithm of the modulus of the reflection coefficient for $k = 0.5$.

In order to distinguish between the acoustic and gravity reflection coefficients it is convenient to introduce the notation K_{RA} and K_{RG}. Although the scalar C_3 in Eqs. (17)–(19) tends to a limit as $\xi_1 \to \infty$, that is, as $\epsilon \to 0$, it is easily shown that K_{RA} and K_{RG} do not approach limiting values. The transition region or reflecting[2] layer shifts toward $z = \infty$ as $\epsilon \to 0$. Thus, the phase of the reflected wave is altered during this process and no limiting value exists. The constant C_3 is invariant since Eqs. (17) and (18) are invariant as $\epsilon \to 0$. The invariance of C_3 implies that $|K_{RG}|$, $\arg K_{RG} + 2\beta \ln(1/\epsilon)$, $|K_{RA}|$, and $\arg K_{RA} - 2\beta \ln(1/\epsilon)$ approach a limit as $\epsilon \to 0$, where β is the dimensionless vertical wave number defined in (9b).

Some of the results of the computations for Case 3 are shown in Figs. 3–7, where the magnitudes of the reflection coefficients are plotted as functions of the vertical wave number β for various values of the horizontal wave number k. For small and large values of k ($k < 0.25$ and $k > 1.0$) it can be seen that the reflection

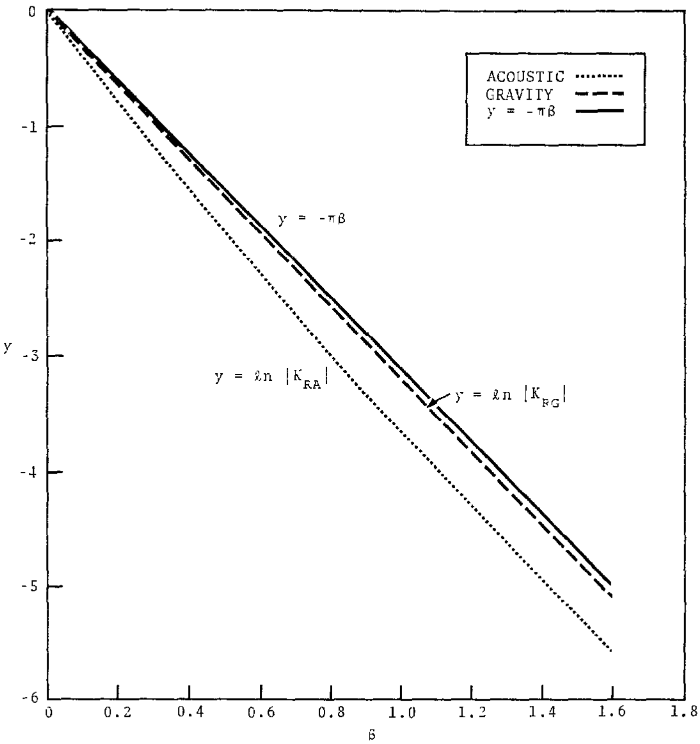

FIG. 6. Logarithm of the modulus of the reflection coefficient for $k = 1.0$.

[2] The reflecting layer is in the vicinity of $\rho(z) = \epsilon$. For a more complete discussion of the reflecting layer and the transition region: see [1, 7].

coefficients are very close to $e^{-\pi\beta}$, which agrees with the results of [1, 2, and 4]. However, for intermediate values of k there is a noticeable deviation from this behaviour (see Figs. 4–6).

For β approximately equal to k it was observed that $\pi\beta - \ln|K_{RG}|$ and $\pi\beta - \ln|K_{RA}|$ achieved a positive local maximum. For most of the calculations[3] $\pi\beta - \ln|K_R|$ was a positive quantity and

$$|K_{RA}(k,\beta)| \leqslant |K_{RG}(k,\beta)| \leqslant e^{-\pi\beta}. \tag{25}$$

It should be noted that the results of the computations depend on the value of γ, and while (25) is a useful summary of the results for $\gamma = 1.4$, it does not hold for all values of γ. It was found, for example, that, for $\gamma = 4.0$, the acoustic wave reflection coefficient is in much better agreement with $e^{-\pi\beta}$ than the gravity wave reflection coefficient.

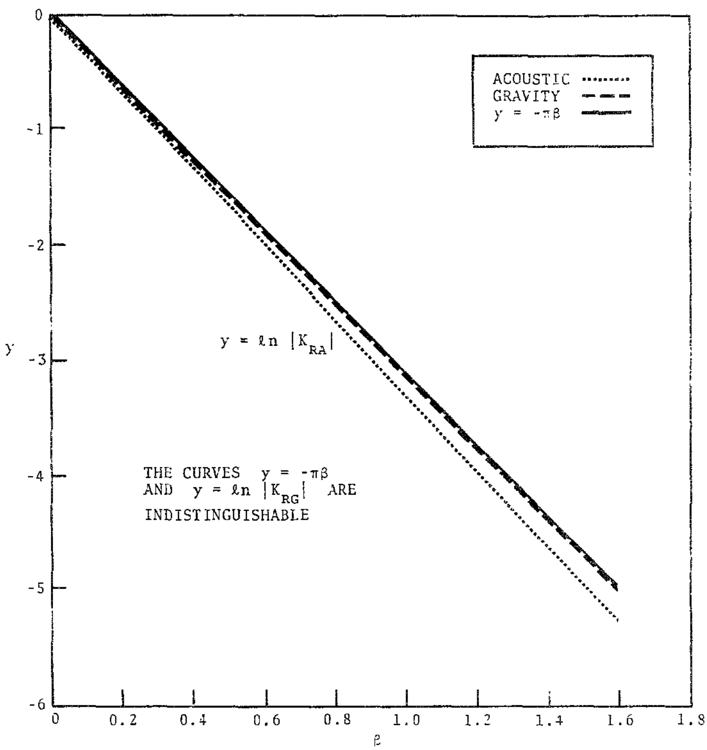

FIG. 7. Logarithm of the modulus of the reflection coefficient for $k = 1.5$.

[3] For $k = 0.005$ (horizontal wavelength of 9000 km), it was found that $|K_{RG}(k,\beta)|$ exceeded $\exp(-\pi\beta)$ slightly. However, $|K_{RG}(k,\beta)| - \exp(-\pi\beta)$ was very small and of the same order as the error for the numerical integration of (13).

It is evident that, in a fluid with an exponentially decreasing density at large altitudes, waves, which in the absence of dissipation would propagate upward, may be reflected downward. The reflection coefficient however, may be sensitive to various features of the fluid model or of the solution. For example, the extreme case of artificial (Rayleigh) viscosity produces no reflection altogether. On the other hand, both viscosity and Newtonian cooling produce the same value for the magnitude of the reflection coefficient for the case of long waves [4], and the present study shows that compressibility has a nonnegligible effect when the horizontal and vertical scales of the motion are of the same order of magnitude. Some solutions for a viscous and thermally conducting isothermal atmosphere (which will be described elsewhere) indicate that for a fairly wide range of Prandtl number the magnitude of the reflection coefficient is determined mostly by viscosity. However, a general conclusion of this type is not warranted without a more complete investigation.

Appendix

About the irregular singularity $\xi = \infty$ it is possible to develop a fundamental set of formal solutions. This can be accomplished, for example, by transforming the differential equation (13) until a guess can be made regarding the structure of the formal solutions. The tactic of transforming (13) is considered in [7, 10]. However, for the viscous problem it is not necessary to obtain a fundamental set of formal solutions. In particular, the boundary layer and transition layer solutions are of secondary importance, and any multiple of these solutions is useful in solving the viscous problem, that is, Eqs. (17)–(19).

It was found that multiples of the boundary layer $\mathbf{y}_+(\xi)$ and transition layer $\mathbf{y}_-(\xi)$ solutions are easily computed without developing the formal expansions with asymptotic scalar growths of $\xi^{1/4} \exp(\pm 2\sigma \sqrt{\xi/\gamma})$. The transition layer solution has dominant exponential growth for a numerical integration of (13) which proceeds in the direction of decreasing $|\xi|$ on $\arg \xi = -\pi/2$. Hence, numerically integrating (13) with a nonzero initial vector over a sufficiently large ξ interval in the direction of decreasing $|\xi|$ results in an accurate approximation of a multiple of the transition layer solution. Similarly, for a numerical integration of (13) in the direction of increasing $|\xi|$ we obtain a multiple of the boundary layer solution.

The formal solutions with algebraic growth in ξ are easily determined by considering the asymptotic developments

$$\hat{\mathbf{y}}(\xi) = \sum_{n=0}^{\infty} \mathbf{a}_n \xi^{-(n+\lambda)}. \tag{A1}$$

Substituting (A1) into (13) yields

$$\lambda^2 - \lambda + \left(\frac{\sigma^2}{\gamma} - k^2 + \frac{\gamma - 1}{\gamma}\frac{k^2}{\sigma^2}\right) = 0. \tag{A2}$$

Relation (A2) is the dispersion relation. The inviscid characteristic growth rates, i.e., λ_1 and λ_2 in (9), satisfy (A2). Aside from a scaling constant, $\xi^{-\lambda}$ corresponds to $\exp(\lambda z)$. Hence, the formal solution (A1) exhibits inviscid growth. In addition, the lead vectors $\mathbf{a}_0$ correspond to the inviscid vector solutions:

$$\mathbf{a}_0 = \begin{bmatrix} 1 \\ \dfrac{\sigma^2/\gamma - k^2}{k(\lambda - 1/\gamma)} \end{bmatrix}, \tag{A3}$$

where $(\sigma^2/\gamma - k^2)$ is nonzero;

$$\mathbf{a}_{0,1} = \begin{bmatrix} -\dfrac{(\lambda_1^2 - \lambda_1\sigma^2/\gamma)}{k(\lambda_1 + 1/\gamma - 3)} \\ 1 \end{bmatrix}, \tag{A4}$$

$$\mathbf{a}_{0,2} = \begin{bmatrix} 1 \\ \dfrac{\sigma^2/\gamma - k^2}{k(\lambda_2 - 1/\gamma)} \end{bmatrix}, \tag{A5}$$

where $(\sigma^2/\gamma - k^2)$ is zero or nearly zero and $\lambda_2 < \frac{1}{2} < \lambda_1$.

Thus, the lead term $\mathbf{a}_0\xi^{-\lambda}$ in the formal expansion (A1) corresponds to an inviscid solution. A similar result is obtained for the limiting case $\lambda_1 = \lambda_2 = \frac{1}{2}$ although (A1) must be modified [7].

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